CERTAIN CASES OF INTEGRABILITY OF THE EQUATIONS OF THE PERTURBED MOTION OF A MATERIAL POINT IN A CENTRAL FORCE FIELD

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We consider here certain cases of motion where we supplement the Newtonian force field with additional forces making the equations of motion integrable in closed form.

1. In deriving the equations of motion we begin from the system

$$d^2\mathbf{r} / dt^2 = -k\mathbf{r} / r^3 + \mathbf{F} \qquad (k = \text{const}, \ r = |\mathbf{r}|) \tag{1.1}$$

with the initial conditions

$$\mathbf{r} = \mathbf{r}_0, \quad d\mathbf{r} / dt = \mathbf{r}_0^{\cdot}, \quad t = t_0 \tag{1.2}$$

Here $\mathbf{r}$ is the position vector of the material point M of unit mass; $\mathbf{F}$ is the force supplementing the Newtonian force. The center of attraction is in the point O.

We consider here the plane motion. The force $\mathbf{F}$ is in the plane of motion. Let us introduce the polar coordinate system r, φ, with the pole at the point O, where φ is the angle measured from a certain stationary direction.

In our polar coordinate system the equations (1.1) take the form [1]

$$r^{\cdot\cdot} - r\varphi^{\cdot 2} = -k/r^2 + F_r, \quad r\varphi^{\cdot\cdot} + 2r^{\cdot}\varphi^{\cdot} = F_\varphi \qquad (r^{\cdot} = dr/dt) \tag{1.3}$$

with the initial conditions

$$r = r_0, \quad r^{\cdot} = r^{\cdot}_0, \quad \varphi = \varphi_0, \quad \varphi^{\cdot} = \varphi_0^{\cdot}, \quad t = t_0 \tag{1.4}$$

Here F_r and F_φ are the components of the force $\mathbf{F}$ along the direction r and φ, respectively. Let us introduce the quantities α and β which are the ratios of the components of $\mathbf{F}$ to the Newtonian force

$$\alpha = F_r r^2 k^{-1}, \quad \beta = F_\varphi r^2 k^{-1} \tag{1.5}$$

We shall change the variables in the equations (1.3) setting

$$u = r^3 \varphi^{\cdot 2} k^{-1} - 1, \qquad v = r^2 r^{\cdot} \varphi^{\cdot} k^{-1} \tag{1.6}$$

and introducing the new independent variable φ .

From (1.3) and (1.6) we obtain for u and v the following system of equations

$$du / d\varphi = -v + 2\beta, \qquad dv / d\varphi = u + \alpha + \beta v / (1 + u) \qquad (1.7)$$

The above system is very convenient if we want to apply approximate asymptotic methods. Let us introduce additional variables x and y

$$x = kr^{-3} \varphi^{\cdot -2}, \qquad y = r^{\cdot} r^{-1} \varphi^{\cdot -1} \qquad (1.8)$$

Using the system (1.3) we find for x and y a system of equations where φ is the independent variable

$$dx/d\varphi = xy - 2\beta x^2, \qquad dy / d\varphi = 1 - x(1 - \alpha) - \beta xy + y^2 \qquad (1.9)$$

The variable y has a simple geometrical meaning, $y = \cot\theta$, where θ is the angle between the vector $\mathbf{r}$ and the velocity $\mathbf{r}^{\cdot}$. The variables u, v and x, y are simply related by

$$u = x^{-1} - 1, \qquad v = yx^{-1} \qquad (1.10)$$

When the systems (1.7) and (1.9) are integrated the solution of (1.3) can be found by quadrature. Let us consider several special cases.

2. Suppose that the components of the supplementary force are

$$F_r = \frac{k}{r^2} \omega \left(\frac{k}{r^3 \varphi^{\cdot 2}}\right), \qquad F_\varphi = m \frac{kr^{\cdot}}{r^3 \varphi^{\cdot}}, \qquad m = \text{const} \qquad (2.1)$$

Here $\omega(x)$ is an arbitrary integrable function of x.

The equations (1.9) have the form

$$\frac{dx}{d\varphi} = xy - 2myx^2, \qquad \frac{dy}{d\varphi} = 1 - x(1 - \omega(x)) + y^2 (1 - mx) \qquad (2.2)$$

and their common integral is

$$\frac{y^2 (1 - 2mx)}{x^2} + \frac{1}{x^2} - \frac{2}{x} - \int \frac{\omega(x)}{x^3} dx = C \qquad (2.3)$$

Let us find a solution of a special case. The equations (2.2) have the particular solution

$$x = 0.5\, m^{-1}, \qquad y = 2g \operatorname{tg} g (\varphi + C_1), \qquad g^2 = 0.25 - 0.125\, m^{-1} (1 - \omega(0.5\, m^{-1})) \qquad (2.4)$$

Integrating (1.8) we find

$$\begin{aligned} r &= C_2 (y^2 + 4g^2) = 4C_2 g^2 \cos^{-2} g (\varphi + C_1) \\ t &= 8g^2 \sqrt{0.5\, C_2^3 k^{-1} m^{-1}} \{0.5 \sin (\varphi + C_1) \cos^{-2} g (\varphi + C_1) + \\ &+ 0.5 \ln \tan(0.5\, g (\varphi + C_1) + 0.25\pi)\} \end{aligned} \qquad (2.5)$$

Motion takes place on a developing spiral possessing an asymptote. We investigate the motion similarly when the right-hand member of g^2 is negative or zero.

The equations (1.3) are integrable in terms of elementary functions in the special case

$$\begin{gathered} m = 0.5, \qquad \omega(x) \equiv 0, \qquad C = -1, \qquad x_0 \neq 1, \qquad y_0 \neq 0 \\ y_0^2 (1 - x_0) + 1 - 2x_0 + x_0^2 = 0 \end{gathered} \qquad (2.6)$$

3. We consider the special case of the supplementary function $\mathbf{F}$

$$F_r = kr^{-2} \omega(\varphi), \qquad F_\varphi = mr\varphi^{\cdot 2}, \qquad m = \text{const} \qquad (3.1)$$

Here $\omega(\varphi)$ is an arbitrary integrable function of φ.

The equations (1.7) become

$$du/d\varphi = 2mu - v + 2m, \qquad dv/d\varphi = u + mv + \omega(\varphi) \tag{3.2}$$

and are integrated in closed form as linear nonhomogeneous equations with constant coefficients. We shall carry out the integration in the simple case

$$\omega(\varphi) \equiv 0, \qquad v = \frac{2m}{1+2m^2}, \qquad u = -\frac{2m^2}{1+2m^2} \tag{3.3}$$

From (1.6) we find the values of r and t in terms of φ

$$d(\ln r) = [v/(1+u)]\, d\varphi = 2m d\varphi, \qquad r = C_1 e^{2m\varphi}$$
$$\varphi'^2 = k(1+u) r^{-3}, \qquad t = \sqrt{C_1^3 (1+2m^2) k^{-1}}\, (3m)^{-1} e^{3m\varphi} + C_2 \tag{3.4}$$

The motion is along a logarithmic spiral. We can prove, in general, that the stationary points of the systems (1.7) or (1.9) correspond to a motion of the point M along a logarithmic spiral.

4. We shall consider the case of the supplementary force

$$F_r = \frac{k(n+1)}{r^2} - r\varphi'^2, \qquad F_\varphi = \frac{km\varphi'}{rr'} \tag{4.1}$$

where the numbers n and m are assumed to be arbitrary. From the equations (1.9) we have

$$\frac{dy}{dx} = \frac{y(x(n-m)+y^2)}{xy^2 - 2mx^2} \tag{4.2}$$

The substitution $u = y^2$ makes the above equation homogeneous. After the integration we obtain the integral of the equation (4.2)

$$x^{-2} y^{-2m/n} (2nx + y^2)^{(m+n)/n} = C_1^2 \tag{4.3}$$

The equations (1.3) are completely integrable in the interesting case

$$m > 0, \qquad n = -m, \qquad x^{-2} y^{-2} = C_1^2, \qquad C_1 > 0, \quad y = C_1 x^{-1} \tag{4.4}$$

The first equation in (1.9) becomes

$$\frac{dx}{d\varphi} = \frac{2m}{C_1}(\lambda^3 - x^3), \qquad \lambda = (C_1^2 (2m)^{-1})^{1/3} \tag{4.5}$$

From (4.5) we find

$$\varphi = \frac{C_1}{2m}\left(\frac{1}{6\lambda^2} \ln \frac{x^2 - \lambda x + \lambda^2}{(x-\lambda)^2} + \frac{1}{\lambda^2 \sqrt{3}} \operatorname{arc\,tan} \frac{2x-\lambda}{\lambda\sqrt{2}}\right) + C_2 \tag{4.6}$$

From equations (4.4) and (4.5) it follows that with increasing φ, x and y approach the constant values

$$x \to \lambda, \qquad y = C_1 \lambda^{-1}, \qquad \varphi \to \infty \tag{4.7}$$

The trajectory of the point M will approach the logarithmic spiral

$$r = C \exp\{C_1 \lambda^{-1} \varphi\}(1 + o(1)) \qquad (\varphi \to \infty) \tag{4.8}$$

Taking x as the independent variable we integrate the equations (1.8)

$$r = C_3 x \frac{1}{(\lambda^3 - x^3)^{1/3}}, \qquad t = \frac{C_1^2 C_3}{3m} \frac{\sqrt{C_3}}{\sqrt{k}} \frac{1}{\sqrt{\lambda^3 - x^3}} + C_4 \tag{4.9}$$

Eliminating φ from (4.6), (4.9) and allowing $x \to \lambda$ we can find C in (4.8)

$$C = C_3 3^{-1/3} \exp(-\pi / 6\sqrt{3}) \tag{4.10}$$

5. If in equations (1.9) the quantities α, β have the form

$$\alpha = 1 + a + by^2x^{-1} - x^{-1}, \quad \beta = my^{-1} \qquad (a, b, m = \text{const}) \tag{5.1}$$

then the equations (1.9) have the integral

$$\left(\frac{y^2}{x}\right)^l \left(2\alpha + (2\beta + 1)\,\frac{y^2}{x}\right)^s \frac{1}{x} = C\,, \qquad l = -\frac{m}{\alpha}\,, \qquad s = \frac{\alpha + m\,(2\beta + 1)}{\alpha\,(2\beta + 1)}$$

If in equations (1.9) we set

$$\alpha = 1 + m^{-1}y^2x^{-1} - x^{-1}, \qquad \beta = \text{const}, \quad m = \text{const} \tag{5.3}$$

then we obtain the integral of the system (1.9)

$$xy^{-2}\,(yx^{-1} + \beta m)^{2+m} = C \tag{5.4}$$

A suitable choice of constants in (5.1) and (5.3) enables us to integrate the equations (1.3) to the very end. For example let $m = -2$ in (5.3). Equations (1.9) then become

$$dx/d\varphi = xy - 2\beta x^2, \qquad dy/d\varphi = -y^2 - \beta xy \tag{5.5}$$

and have the obvious solution

$$x = [2\beta\varphi + C_1]^{-1}, \qquad y = 0 \tag{5.6}$$

From (1.8) we find r and t

$$r = r_0 = \text{const}, \qquad t = \sqrt{r_0^3k^{-1}}\,\beta^{-1}\,(2\beta\varphi + C_1) \tag{5.7}$$

Motion takes place on a circular orbit. The radial component of the supplementary force F_r equals

$$F_r = kr_0^{-2}\,(1 - C_1 - 2\beta\varphi) \tag{5.8}$$

6. We consider here the particular case of the supplementary force $\mathbf{F}$

$$F_r = \frac{k}{r^2}\,\omega\,(u, v), \qquad F_\varphi = \frac{k}{2r^2}\,v \tag{6.1}$$

where u and v are determined by (1.6), and $\omega\,(u, v)$ is an arbitrary function of u and v. The equations (1.7) can be integrated in closed form. We shall find the solution in the special case

$$F_r = \alpha kr^{-2}, \qquad F_\varphi = {}^1/_2\, r^{\cdot}\varphi^{\cdot}, \qquad \alpha = \text{const} \tag{6.2}$$

From (1.7) we find (6.3)

$$u = C_1, \quad v = a\tan(b\varphi + C_2); \quad a^2 = 2\,(C_1 + \alpha)\,(1 + C_1) > 0, \quad b = 0.5\,a\,(1 + C_1)^{-1}$$

From (1.6) we find the expression for r

$$d\,(\ln r) = [v\,/\,(1 + u)]\,d\varphi, \qquad r = C_3\cos^{-2}(b\varphi + C_2) \tag{6.4}$$

This trajectory coincides with the trajectory given by formula (2.6). The time t is determined by a formula similar to (2.5). Motion takes place on a developing spiral. Besides, for F_φ we can find an expression in terms of φ

$$F_\varphi = 0.5\,ka\,C_3^{-2}\sin\,(b\varphi + C_2)\cos^3(b\varphi + C_3) \tag{6.5}$$

Approaching the asymptote, when $b\varphi + C_2 \to {}^1/_2\,\pi$, the component $F_\varphi \to 0$.

7. Here, the dependent variables in (1.3) will be

$$u = r^{-1}, \qquad p = r^4\varphi^{\cdot 2} \tag{7.1}$$

and the argument is the angle ϕ.

We find the equations for u and p which coincide with the equations (2.2) in reference [2]

$$\frac{dp}{d\varphi} = \frac{2}{u^3}\,F_\varphi, \qquad \frac{d^2u}{d\varphi^2} + u = \frac{k}{p} - \frac{1}{u^2p}\,F_r - \frac{1}{u^3p}\,F_\varphi\,\frac{du}{d\varphi} \tag{7.2}$$

Further on new cases of integrability of equations (7.2) are described.

8. Let the supplementary force be in the form

$$F_r = \gamma(\varphi)\, r^{-3}, \qquad F_\varphi = \delta(\varphi)\, r^{-3} \tag{8.1}$$

where the functions $\gamma(\varphi)$ and $\delta(\varphi)$ are related by

$$\gamma(\varphi) = \frac{d\delta(\varphi)}{d\varphi} - 2\frac{\delta^2(\varphi)}{p(\varphi)} - p(\varphi), \qquad p(\varphi) = 2\int \delta(\varphi)\, d\varphi \tag{8.2}$$

The second equation in (7.2) takes on the integrable form

$$\frac{d}{d\varphi}\left[\frac{du}{d\varphi} + \frac{\delta(\varphi)}{p(\varphi)}u\right] = \frac{k}{p(\varphi)} \tag{8.3}$$

Consequently, the system (7.2) is integrable in closed form. If $p(\varphi)$ in (7.2) is assumed to be prescribed and if we set $F_r = \gamma(\varphi)\, r^{-3}$, and $F_\varphi = 0.5\, r^{-3}\, dp / d\varphi$, then the second equation in (7.2) becomes

$$\frac{d^2u}{d\varphi^2} + \frac{1}{2}\frac{du}{d\varphi}\left(\frac{d\ln p}{d\varphi}\right) + u = \frac{k - \gamma(\varphi)}{p(\varphi)} \tag{8.4}$$

We can choose a suitable function $p(\varphi)$ to make the equation (8.4) integrable.

By selecting suitable arbitrary functions and constants which appear in the expression for the supplementary force, we can use the cases already investigated to examine new cases with different supplementary forces and investigate various motions approximately.

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